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The Kepler problem in Dirac theory for a particle with position-dependent mass

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Abstract

An exact solution of the Dirac equation for a particle whose potential energy and mass are inversely proportional to the distance from the force centre has been found. The bound states exist provided the length scale a which appears in the expression for the mass is smaller than the classical electron radius e^2/mc^2. Furthermore, bound states also exist for negative values of a even in the absence of the Coulomb interaction. In first order perturbation theory the corrections to the energy caused by the terms of dipole nature in the expression for the mass are calculated. Quasirelativistic expansion of the energy has been carried out, and a modified expression for the fine structure of the energy levels has been obtained. The problem of kinetic energy operator in the Schrödinger equation is discussed for the case of position-dependent mass. In particular, we have found that for highly excited states the mutual ordering of the inverse mass and momentum operator in the non-relativistic theory is not important.

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1. Introduction

The concept of the effective mass in theoretical physics is quite efficient because it allows us to reduce a many-body problem to a single-particle one, without the loss of the main contributions to the mechanism of the formation of various physical phenomena from the inter-particle interactions. We can exemplify this statement by specific problems from semiconductor physics, superfluid ^{4}He theories, problems of nanophysics as well as by a number of other problems from condensed matter physics [1–5]. At the same time, the suggested approach raises the issue of mutual ordering of the momentum operator in the Schrödinger equation and the inverse effective mass in the kinetic energy operator, which is the momentum function of the particle coordinate [6–11]. This problem seems to disappear if one proceeds from the Dirac equation, but the transition to the non-relativistic case is far from being simple as it might look at first glance [8].

In this paper we suggest a solution of the Kepler problem (i.e., a study of the particle's movement in the Coulomb potential) in Dirac theory for a particle with a given effective mass m^* dependent on coordinate $\mathbf{r}$. We assume this dependence to be relevant at distances of the particle from the force centre smaller than the Compton length $\lambda = \hbar/mc$, where m is the mass m^* at $r \to \infty$. For such distances the notion of the particle coordinate is lost as attempts to localize the particle in the space domain with linear dimensions $\sim\lambda$ lead to the creation of new particles. In this connection, it seems possible to try and take effectively into account the processes of the interaction of the particles with the force centre at subatomic scales through a coordinate-dependent mass.

In the general case, we assume that, e.g., for an electron the value of m^* can be presented in the form of a multipole expansion

$$m^* = m\left(1 + \frac{a}{r} + \frac{\mathbf{br}}{r^3} + \cdots\right), \tag{1.1}$$

where $a, \mathbf{b}, \ldots$ are constants formed by the mechanism of the particle interaction with the vacuum fluctuations in the presence of the force centre; later on we consider these constants to be the given initial parameters of the problem. The processes of the interaction at subatomic scales within quantum field theory lead, in particular, to the deformation of the Coulomb interaction in the atom when the distance between the electron and the nucleus is small. In other words, it is quite possible to consider the electron charge as a function of the particle coordinate. We can therefore speak about the effective consideration, in Dirac theory, of the radiation effects which are due to the renormalization of the electron mass and charge; that is why we can make an attempt to account, within such a phenomenological approach, for the observable superfine structure of the atom energy spectrum.

The above-mentioned arguments can be disregarded. Then, one treats the Dirac equation formally as a mathematical problem in which the mass m^* is dependent on the radius vector $\mathbf{r}$.

Below, we provide an exact solution of the Kepler problem in Dirac theory for the case when only the first two terms are taken into account in the expansion for the effective mass (1.1). Such a problem was considered in [12] within another problem of the so-called mixed vector–scalar potentials in the Dirac equation. We propose an approach differing from that of [12]. Additionally, the dipole term in (1.1) is taken into account by means of perturbation theory. We investigate also the non-relativistic contribution of the position-dependent mass to the energy levels.

2. Initial equations

We start from the Dirac equation (in familiar notation), i.e.,

$$[(\hat{\boldsymbol{\alpha}}\hat{\mathbf{p}})c + m^*c^2\hat{\beta} + U]\psi = E\psi, \tag{2.1}$$

where $\hat{\boldsymbol{\alpha}}$, $\hat{\beta}$ are the Dirac matrices, the particle effective mass is

$$m^* = m\left[1 + \frac{a}{r} + (\hat{\boldsymbol{\sigma}}\mathbf{n})\varphi\right], \tag{2.2}$$

$\varphi = \varphi(r)$ and the energy of the Coulomb interaction is

$$U = -\frac{e^2}{r}, \tag{2.3}$$

where e is the particle charge, and $\hat{\boldsymbol{\sigma}}$ means a 4×4 matrix:

$$\hat{\boldsymbol{\sigma}} \to \begin{pmatrix} \hat{\boldsymbol{\sigma}} & 0 \\ 0 & \hat{\boldsymbol{\sigma}} \end{pmatrix},$$

where the diagonal items are the usual Pauli matrices, $\mathbf{n} = \mathbf{r}/r$ is a unit vector. Since the spin is the only vector characterizing a particle, it is natural to assume that in equation (1.1) the vector $\mathbf{b} \sim \boldsymbol{\sigma}$. In addition, since $(\hat{\boldsymbol{\sigma}}\mathbf{n})^2 = 1$, the higher powers of $\hat{\boldsymbol{\sigma}}$ are absent in (1.1). This fact explains the dependence (2.2).

In the following we calculate the energy spectrum E. To do this, we introduce the function $\bar{\psi}$ so that

$$\psi = [(\hat{\boldsymbol{\alpha}}\hat{\mathbf{p}})c + m^* c^2 \hat{\beta} + (E - U)]\bar{\psi}, \tag{2.4}$$

that is, we apply the substitution used also in [13–15]. For the new function $\bar{\psi}$ from the Dirac equation (2.1) we find

$$\{[(\hat{\boldsymbol{\alpha}}\hat{\mathbf{p}})c + m^* c^2 \hat{\beta}]^2 - c[(\hat{\boldsymbol{\alpha}}\hat{\mathbf{p}})U - U(\hat{\boldsymbol{\alpha}}\hat{\mathbf{p}})] - (E - U)^2\}\bar{\psi} = 0. \tag{2.5}$$

It is easy to see that the commutator

$$[\hat{\boldsymbol{\alpha}}\hat{\mathbf{p}}, U] = -\mathrm{i}\hbar(\boldsymbol{\sigma}\mathbf{n})\hat{\beta}'\frac{\mathrm{d}U}{\mathrm{d}r},$$

and

$$[(\hat{\boldsymbol{\alpha}}\hat{\mathbf{p}})c + m^* c^2 \beta]^2 = \hat{\mathbf{p}}^2 c^2 + m^{*2} c^4 - \mathrm{i}\hbar c^3 (\boldsymbol{\sigma}\nabla m^*)\hat{\beta}'', \tag{2.6}$$

where

$$\hat{\beta}' = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad \hat{\beta}'' = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \tag{2.7}$$

are 4×4 matrices.

Our initial equation (2.5) now takes the following form:

$$\left\{\hat{\mathbf{p}}^2 c^2 + m^{*2} c^4 - \mathrm{i}\hbar c^3 (\boldsymbol{\sigma}\nabla m^*)\hat{\beta}'' + \mathrm{i}\hbar c(\boldsymbol{\sigma}\mathbf{n})\hat{\beta}'\frac{\mathrm{d}U}{\mathrm{d}r} - (E - U)^2\right\}\bar{\psi} = 0. \tag{2.8}$$

Using explicit expressions for m^* and U (see equations (2.2), (2.3)), we obtain, after simple calculations,

$$\left\{\frac{\hat{\mathbf{p}}^2}{2m} - \left(\frac{E}{mc^2}e^2 - mc^2 a\right)\frac{1}{r} + \frac{1}{2mr^2}\left[\frac{\mathrm{i}\hbar}{c}(\boldsymbol{\sigma}\mathbf{n})(mc^2 a\hat{\beta}'' + e^2\hat{\beta}') + m^2 c^2 a^2 - \frac{e^4}{c^2} + \hat{W}\right]\right\}\bar{\psi}$$
$$= \left(\frac{E^2 - m^2 c^4}{2mc^2}\right)\bar{\psi}, \tag{2.9}$$

where

$$\hat{W} = \frac{mc^2}{2}\varphi^2 + mc^2\varphi\left(1 + \frac{a}{r}\right)(\boldsymbol{\sigma}\mathbf{n}) - \frac{\mathrm{i}\hbar c}{2r^2}\frac{\mathrm{d}}{\mathrm{d}r}(r^2\varphi)\hat{\beta}''. \tag{2.10}$$

We note that our equation has the form of the Schrödinger equation for a particle moving in the Coulomb potential with an addition to the centrifugal energy. Operator $\hat{W}$ is considered as a perturbation.

3. The radial equation

We now pass to spherical coordinates in equation (2.9):

$$\left\{-\frac{\hbar^2}{2m}\frac{1}{r}\frac{\mathrm{d}^2}{\mathrm{d}r^2}r + \frac{1}{2mr^2}\left[\hat{\mathbf{L}}^2 + \frac{\mathrm{i}\hbar}{c}(\boldsymbol{\sigma}\mathbf{n})(mc^2 a\hat{\beta}'' + e^2\hat{\beta}') + m^2 c^2 a^2 - \frac{e^4}{c^2}\right]\right.$$
$$\left. - \left(\frac{Ee^2}{mc^2} - mc^2 a\right)\frac{1}{r} + \hat{W}\right\}\bar{\psi} = \frac{E^2 - m^2 c^4}{2mc^2}\bar{\psi}, \tag{3.1}$$

where $\hat{\mathbf{L}}$ is the angular momentum operator.

As the operator in square brackets depends solely on the angles, the variables in equation (3.1) can be separated. We further make use of the fact that

$$\hat{\mathbf{L}}^2 = (\boldsymbol{\sigma}\hat{\mathbf{L}})[(\boldsymbol{\sigma}\hat{\mathbf{L}}) + \hbar],$$

and introduce the operator

$$\hat{\Lambda} = -[(\boldsymbol{\sigma}\hat{\mathbf{L}}) + \hbar] + \frac{\mathrm{i}}{c}(\boldsymbol{\sigma}\mathbf{n})(mc^2 a\hat{\beta}'' + e^2\hat{\beta}'), \tag{3.2}$$

with

$$\hat{\Lambda}^2 = [(\boldsymbol{\sigma}\hat{\mathbf{L}}) + \hbar]^2 + m^2c^2a^2 - e^4/c^2.$$

We have used here the fact that

$$[\hat{\mathbf{L}}\mathbf{n}] + [\mathbf{n}\hat{\mathbf{L}}] = 2\mathrm{i}\hbar\mathbf{n}.$$

It is now not difficult to show that the operator in square brackets in equation (3.1) equals $\hat{\Lambda}(\hat{\Lambda} + \hbar)$. Then, we rewrite equation (3.1) as follows:

$$\left\{-\frac{\hbar^2}{2m}\frac{1}{r}\frac{\mathrm{d}^2}{\mathrm{d}r^2}r + \frac{\hat{\Lambda}(\hat{\Lambda} + \hbar)}{2mr^2} - \left(\frac{E}{mc^2}e^2 - mc^2a\right)\frac{1}{r} + \hat{W}\right\}\bar{\psi} = \frac{E^2 - m^2c^4}{2mc^2}\bar{\psi}. \tag{3.3}$$

As $(\hat{\boldsymbol{\sigma}}\hat{\mathbf{L}}) = (\hat{\mathbf{J}}^2 - \hat{\mathbf{L}}^2 - \hat{\mathbf{S}}^2)/\hbar$, where the total angular momentum $\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{S}}$, and the spin operator $\hat{\mathbf{S}} = \hbar\hat{\boldsymbol{\sigma}}/2$, it is easy to see that the operator

$$\hat{\Lambda}^2 = \left(\frac{\hat{\mathbf{J}}^2 - \hat{\mathbf{L}}^2 - \hat{\mathbf{S}}^2}{\hbar} + \hbar\right)^2 + (mca)^2 - \left(\frac{e^2}{c}\right)^2.$$

It then follows that the eigenvalues of this operator

$$\Lambda^2 = \hbar^2\left[j(j+1) - l(l+1) + \frac{1}{4}\right]^2 + (mca)^2 - \left(\frac{e^2}{c}\right)^2,$$
$$j = l \pm 1/2, \qquad l = 0, 1, 2, \ldots.$$

Further, if $j = l + 1/2$, then

$$\Lambda^2 = \hbar^2(l+1)^2 + (mca)^2 - (e^2/c)^2,$$

and if $j = l - 1/2$, then

$$\Lambda^2 = \hbar^2 l^2 + (mca)^2 - (e^2/c)^2.$$

Now it is not difficult to find the eigenvalues of the operator $\hat{\Lambda}$ (3.2):

$$\Lambda = -\hbar\sqrt{(l+1)^2 + (mca/\hbar)^2 - (e^2/\hbar c)^2},$$

for $j = l + 1/2$ and

$$\Lambda = \hbar\sqrt{l^2 + (mca/\hbar)^2 - (e^2/\hbar c)^2},$$

for $j = l - 1/2$. Finally, we can easily find the eigenvalues of the operator $\hat{\Lambda}(\hat{\Lambda} + \hbar)$:

$$\Lambda(\Lambda + \hbar) = \hbar^2 l^*(l^* + 1),$$

where the quantum number

$$l^* = \sqrt{(j + 1/2)^2 + (mca/\hbar)^2 - (e^2/\hbar c)^2} - 1/2 \mp 1/2. \tag{3.4}$$

In equation (3.4), the upper sign is for $j = l + 1/2$ while the lower sign is for $j = l - 1/2$.

If the perturbation operator is not taken into account, $\hat{W} = 0$, substituting the operator $\hat{\Lambda}(\hat{\Lambda} + \hbar)$ with its eigenvalues in equation (3.3), we can write down the equation for the radial part R of $\bar{\psi}$:

$$\left\{-\frac{\hbar^2}{2m}\frac{1}{r}\frac{\mathrm{d}^2}{\mathrm{d}r^2}r + \frac{\hbar^2 l^*(l^*+1)}{2mr^2} - \left(\frac{E}{mc^2}e^2 - mc^2a\right)\frac{1}{r}\right\}R = \frac{E^2 - m^2c^4}{2mc^2}R. \tag{3.5}$$

4. Energy eigenvalues: discussion of the results

Formally, equation (3.5) coincides with the non-relativistic Schrödinger equation for the Kepler problem with the energy

$$E^* = \frac{E^2 - m^2c^4}{2mc^2}$$

and the charge squared

$$e^{*2} = \frac{E}{mc^2}e^2 - mc^2a. \tag{4.1}$$

For the existence of the bound states, it is necessary that the 'potential energy' in equation (3.5) should be of the attractive nature or, otherwise said, the charge squared e^{*2} should be a positive value $e^{*2} > 0$.

Then one can write the Bohr formula for the energy levels E^*:

$$E^* = -\frac{me^{*4}}{2\hbar^2(n_r + l^* + 1)^2}, \tag{4.2}$$

where $n_r = 0, 1, 2, \ldots$ is the radial quantum number, hence, the equation for E

$$\frac{E^2 - m^2c^4}{2mc^2} = -\frac{m(Ee^2/mc^2 - mc^2a)^2}{2\hbar^2 n^{*2}},$$

where we introduced the 'principal' quantum number

$$n^* = n_r + l^* + 1 = n_r + \sqrt{(j + 1/2)^2 + (a/\lambda)^2 - \alpha^2} + 1/2 \mp 1/2,$$

where $\alpha = e^2/\hbar c$ is the fine structure constant and $\lambda = \hbar/mc$ is the Compton length.

Solving this equation, we finally find for the energy spectrum

$$E = \frac{mc^2}{1 + (\alpha/n^*)^2}\left[\frac{a}{\lambda}\frac{\alpha}{n^{*2}} + \sqrt{1 + \frac{\alpha^2 - (a/\lambda)^2}{n^{*2}}}\right]. \tag{4.3}$$

The wavefunctions R are the usual radial functions of the non-relativistic hydrogen problem with the quantum numbers n^*, l^* and with the charge squared e^{*2}. In order to determine the full wavefunction ψ, it is necessary to substitute the function $\bar{\psi}$ into equation (2.4) in the form of the product of the radial function R and the spherical spinor.

Within another problem of vector–scalar potentials in the Dirac equation, expression (4.3) was first found in [12] using a usual reduction of the equation to a set of two equations followed then by a standard expansion of radial functions into powers of r. We have solved the problem by another method, namely, reducing it to a formally non-relativistic one with respectively renormalized physical quantities and quantum numbers. This method, in our opinion, provides the possibility of solving the eigenvalue problem with more complicated dependences $m^* = m^*(\mathbf{r})$.

The bound states exist when $e^{*2} > 0$. For $a < 0$ this condition, as seen from (4.1), always holds true. For $a > 0$, using in equation (4.1) the energy E from equation (4.3), we find the condition for the parameter a that appears not to depend on the quantum number n^*. The result is

$$a < e^2/mc^2.$$

Thus the bound states exist for arbitrary negative values of the parameter a. If the length parameter $a > 0$, it should be smaller than the electron classical radius. In other words, these are the distances $1/\alpha \backsimeq 137$-fold smaller than the Compton length λ, where the well-defined

notion of the particle coordinate is lost. It is interesting to note that for $a = e^2/mc^2$ the energy $E = mc^2$, hence, there is just one level.

On the other hand, if the parameter $a = 0$, then from equation (4.3) we get the well-known formula for a fine structure of the hydrogen atom energy spectrum, for which there is the exact solution of the Dirac equation.

In the absence of the Coulomb interaction ($e^2 = 0$), the bound states exist for $a < 0$ with the energy levels

$$E = mc^2\sqrt{1 - \left(\frac{a}{\lambda n^*}\right)^2},$$

where the quantum number l^*, contained in n^*, is determined by equation (3.4) for $e^2 = 0$.

For the ground state, for $n^* = \sqrt{1 + (a/\lambda)^2 - \alpha^2}$, one obtains from equation (4.3)

$$E = \frac{mc^2}{1 + (a/\lambda)^2}\left[\frac{a}{\lambda}\alpha + \sqrt{1 + (a/\lambda)^2 - \alpha^2}\right].$$

For $\alpha = 0$ the ground state energy ($a < 0$)

$$E = \frac{mc^2}{\sqrt{1 + (a/\lambda)^2}}.$$

This quantity plays the role of the particle rest energy; thus, the mean value of the effective mass, i.e.,

$$m^* = \frac{m}{\sqrt{1 + (a/\lambda)^2}}$$

is smaller than the mass m, which corresponds to the negative parameter a. If parameter $|a| \gg \lambda$, then the ground state energy $E = \hbar c/|a|$ is of the nature of the Casimir energy concentrated in the volume $\sim|a|^3$.

Let us now consider the non-relativistic limit $c \to \infty$. It is important for the revealing of the mechanism of the transition to the non-relativistic description of the particle with a position-dependent mass. We assume that the natural length scale for the quantity a is the classical radius of the electron, which, in our problem, is the upper limit for a; hence we put $a = \bar{a}e^2/mc^2$, $\bar{a} < 1$. We believe that the numeric value of $\bar{a}$ does not depend on any fundamental constants, and that, ultimately, this value is the initial characteristic of the particle. Now the formula for the energy takes the form

$$\begin{aligned} E &= \frac{mc^2}{1 + (\alpha/n^*)^2}\left[\left(\frac{\alpha}{n^*}\right)^2\bar{a} + \sqrt{1 + \left(\frac{\alpha}{n^*}\right)^2(1 - \bar{a}^2)}\right], \\ n^* &= n_r + \sqrt{\left(j + \frac{1}{2}\right)^2 + \alpha^2(\bar{a}^2 - 1)} + \frac{1}{2} \mp \frac{1}{2}. \end{aligned} \tag{4.4}$$

We expand this expression in power series over α up to α^2:

$$E = mc^2 - \frac{me^4}{2\hbar^2 n^2}(1 - \bar{a})^2 - \frac{me^4}{2\hbar^2 n^4}\alpha^2(1 - \bar{a})^3\left[\frac{n}{j + 1/2}(1 + \bar{a}) - \frac{3}{4}(1 + \bar{a}/3)\right], \tag{4.5}$$

where $n = n_r + l + 1$ is the principal quantum number. For $\bar{a} = 0$ we have the familiar fine structure formula. As we can see, the dependence of the mass on the coordinate also deforms the non-relativistic term (i.e., the Bohr formula) in such a way as if the particles mass had been substituted with $m(1-\bar{a})^2$. For $\bar{a} > 0$ this effect has the same sign as the correction accounting for the finiteness of the nucleus mass. The degeneracy of terms, in particular, $S_{1/2}$ and $P_{1/2}$,

holds because we have the specific dependence of the mass on r. If, in expansion (1.1), we also left the next terms, this degeneracy would be removed.

Let us calculate the correction of the perturbation operator $\hat{W}$ into the energy levels (4.3). To do this, we refer to equation (3.3), and in the first order of the perturbation theory instead of (4.2), one has

$$E^* = -\frac{me^{*4}}{2\hbar^2 n^{*2}} + \langle \hat{W} \rangle,$$

where the angle brackets denote the averaging over the wavefunctions of the non-perturbed problem. In this way, we arrive again at the equation for the energy E. Substituting the energy in the form $E = E^{(0)} + E^{(1)}$ into equation (4.6), where $E^{(0)}$ is given by (4.3), we find for the first-order correction $E^{(1)}$ the following:

$$E^{(1)} = \langle \hat{W} \rangle \Bigg/ \sqrt{1 + \frac{\alpha^2 - (a/\lambda)^2}{n^{*2}}}. \tag{4.6}$$

If the function φ is of a nature given by equation (1.1),

$$\varphi = \frac{b^2}{r^2},$$

$b = \text{const}$, then the last term in equation (2.10) for the operator $\hat{W}$ equals zero and the mean value of the second term in $\hat{W}$ also equals zero. Therefore, the contribution into $E^{(1)}$ is formed solely by the first term in the expression for $\hat{W}$. Using the mean value of $1/r^4$ averaged over the hydrogen wavefunctions [16], we find the explicit form of the correction to the energy ($l^* > 1/2$)

$$E^{(1)} = \frac{me^4}{2\hbar^2}\bar{b}^4\alpha^2 \frac{\left(\sqrt{1+\alpha^2(1-\bar{a}^2)/n^{*2}} - \bar{a}\right)^4}{(1+\alpha^2/n^{*2})^4\sqrt{1+\alpha^2(1-\bar{a}^2)/n^{*2}}} \frac{3n^{*2} - l^*(l^*+1)}{2n^{*5}(l^*+3/2)(l^*+1)l^*(l^{*2}-1/4)}, \tag{4.7}$$

where $\bar{b} = b/\lambda$. Therefore, the degeneracy with respect to the orbital quantum number l still holds in first order perturbation theory. It disappears only in second order perturbation theory due to the second term in $\hat{W}$.

5. Conclusion

We have found the exact solution of the Dirac equation for a particle with position-dependent mass, which might be useful in the study of the corresponding non-relativistic problem as a reference result. The next terms, which we have neglected in this work (in particular the dipolar one) and which are responsible for the super-fine structure of the energy spectrum, can be taken into account by means of standard perturbation theory.

Let us return to the ordering problem in the kinetic energy operator in the Schrödinger equation when the mass of the particle depends on the coordinate. We have already seen that the corrections from this dependence for the non-relativistic limit are due to the fact that in the mass this dependence happens to be $\sim 1/c^2$. The mechanism of the appearance of the above-said correction is quite simple. The particle rest energy is $m^*c^2 = mc^2(1 + \bar{a}e^2/mc^2r) = mc^2 + \bar{a}e^2/r$. In the linear approximation, according to perturbations theory, this energy is $(mc^2 + \bar{a}\langle e^2/r\rangle) = mc^2 + \bar{a}e^2/a_{\mathrm{B}}n^2 = mc^2 + \bar{a}me^4/\hbar^2n^2$, where $a_{\mathrm{B}} = \hbar^2/me^2$ is the Bohr radius. This expression, together with the zeroth approximation for E, in accordance with the Bohr formula, yields $mc^2 - me^4(1 - 2\bar{a})/2\hbar^2n^2$, which, in the linear approximation in $\bar{a}$, coincides with formula (4.5). In other words, only on condition that the dependence of the

mass on the coordinate is $\sim 1/c^2$, do we deform the non-relativistic expression for the energy levels. At the same time, it is irrelevant how we order the momentum operator with the inverse mass in the kinetic energy operator because the corrections are of the order of $1/c^2$.

Thus, the issue of the form of the kinetic energy in the Schrödinger equation for position-dependent mass has, in the general case, quite a limited sense. If this dependence arises in the non-relativistic case, as a result of the reduction of the many-particle problem to a single-particle one, or as problems in curved space, the specific ordering of the operators appears rather naturally in each problem. As a consequence, the energy levels, which depend on some ordering parameter, differ from problem to problem. This has been shown in [17], where the Schrödinger equation with the Coulomb potential is solved with some other dependence of the mass on the radial variable. Yet it is important that for highly excited states, i.e., for large values of quantum numbers, the dependence of the energy on the ordering 'parameter' is, probably, insignificant, and the energy can be found using the quasiclassical Bohr–Sommerfeld quantization method. For instance, for the dependence of m^* on r given by equation (2.2) the direct solution of the Schrödinger equation, in which $1/\sqrt{m^*}$ stands leftwards and rightwards of the momentum square, obviously yields the same energy levels as the Bohr–Sommerfeld method. Finally, there is also another argument in support of our theory, which follows from the results of [17], that for large quantum numbers the energy does not depend on the inverse mass and momentum ordering 'parameter'.

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